ON SOME CRITICAL PROBLEMS FOR THE FRACTIONAL LAPLACIAN OPERATOR

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Abstract. We study the effect of lower order perturbations in the existence of positive solutions to the following critical elliptic problem involving the fractional Laplacian:

$$\begin{cases} (-\Delta)^{\alpha/2}u = \lambda u^q + u^{\frac{N+\alpha}{N-\alpha}}, \quad u>0 & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geq 1$, $\lambda > 0$, $0 < q < \frac{N+\alpha}{N-\alpha}$, $0 < \alpha < \min\{N, 2\}$. For suitable conditions on α depending on q, we prove: In the case $q < 1$, there exist at least two solutions for every $0 < \lambda < \Lambda$ and some $\Lambda > 0$, at least one if $\lambda = \Lambda$, no solution if $\lambda > \Lambda$. For $q = 1$ we show existence of at least one solution for $0 < \lambda < \lambda_1$ and nonexistence for $\lambda \geq \lambda_1$. When $q > 1$ the existence is shown for every $\lambda > 0$. Also we prove that the solutions are bounded and regular.

1. Introduction

Problems of the type

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

for different kind of nonlinearities f, have been the main subject of investigation in a large amount of works in the last thirty years. See for example the list (far from complete) [1, 2, 9, 20]. One of the most important cases of problem (1.1) is the critical power $f(u) = u^{\frac{N+2}{N-2}}$, $N > 2$, since it is well known that this problem has no positive solution provided the domain is starshaped. In a pioneering work [9], Brezis and Nirenberg showed that, contrary to intuition, the critical problem with small linear perturbations can provide positive solutions. After that, in [2], using the results on concentration-compactness of Lions, [20], Ambrosetti, Brezis and Cerami proved some results on existence and multiplicity of solutions for a sublinear perturbation of the critical power, among others.

Recently, several studies have been performed for classical elliptic equations with the Laplacian operator substituted by its fractional powers. In particular, in [23] it is studied the problem

$$\begin{cases} (-\Delta)^{1/2}u = \lambda u + u^{\frac{N+1}{N-1}} & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

the analogue case to the problem in [9], but with the square root of the Laplacian instead of the Laplacian. This operator is defined in [12] through the spectral decomposition of the Laplacian operator in Ω with zero Dirichlet boundary conditions. Prior to this study, in [12] the authors proved that there is no solution in the case $\lambda = 0$ and Ω starshaped.

In this paper we are interested in the following perturbations of the critical power case for different powers of the Laplacian,

$$(P_\lambda) \quad \begin{cases} (-\Delta)^{\alpha/2}u = f_\lambda(u) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases}$$

with $f_\lambda(u) := \lambda u^q + u^{\frac{N+\alpha}{N-\alpha}}$, $0 < q < \frac{N+\alpha}{N-\alpha}$, $0 < \alpha < 2$ and $N > \alpha$. Along the paper we will look only for positive solutions to (P_λ) (so many times we will omit the term "positive").

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For the definition of the fractional Laplacian operator we follow some ideas of [12], together with results from [5] and [13]. In particular, we define the eigenvalues λ_j of $(-\Delta)^{\alpha/2}$ as the power $\alpha/2$ of the eigenvalues ρ_j of $(-\Delta)$, i.e., $\lambda_j = \rho_j^{\alpha/2}$; both with zero Dirichlet boundary data. With this definition, it has been proved in [5], using a generalized Pohozaev identity, that problem (P_λ) has no solution for $\lambda = 0$ whenever Ω is a starshaped domain.

Our main results dealing with Problem (P_λ) are the following.

Theorem 1.1. *Let $0 < q < 1$. Then, there exists $0 < \Lambda < \infty$ such that the problem (P_λ)*

(1) *has no positive solution for $\lambda > \Lambda$;*
(2) *has a minimal positive solution for any $0 < \lambda \leq \Lambda$. Moreover the family of minimal solutions is increasing with respect to λ;*
(3) *if $\lambda = \Lambda$ there is at least one positive solution;*
(4) *if $\alpha \geq 1$ there are at least two positive solutions for $0 < \lambda < \Lambda$.*

Theorem 1.2. *Let $q = 1$, $0 < \alpha < 2$ and $N \geq 2\alpha$. Then the problem (P_λ)*

(1) *has no positive solution for $\lambda \geq \lambda_1$;*
(2) *has at least one positive solution for each $0 < \lambda < \lambda_1$.*

Theorem 1.3. *Let $1 < q < \frac{N+\alpha}{N-\alpha}$, $0 < \alpha < 2$ and $N > \alpha(1 + 1/q)$. Then the problem (P_λ) has at least one positive solution for any $\lambda > 0$.*

The restriction $\alpha \geq 1$ in Theorem 1.1-(4) seems to be technical. We remember that in the study of the corresponding subcritical case performed in [5] the same restriction on α appeared. In that case the difficulty was to find a Liouville-type theorem for $0 < \alpha < 1$. Here, due to the lack of regularity, see Proposition 5.2, it is not clear how to separate the solutions in the appropriate way, Lemma 3.2, see also [15, 16].

On the other hand, we have left open the range $\alpha < N < 2\alpha$ in Theorem 1.2. See the special case $\alpha = 2$ and $N = 3$ in [9]. If $\alpha = 1$ this range is empty, see [23].

As to the regularity of solutions, they are bounded and "classical" in the sense that they have as much regularity as it is required in the equation, i.e., they possess α "derivatives", see Propositions 5.1 and 5.2. Even more, if $\alpha = 1$, they belong to $\mathcal{C}^{1,q}(\overline{\Omega})$ or $\mathcal{C}^{\infty}(\overline{\Omega})$, whenever $0 < q < 1$ or $q \geq 1$, respectively.

Organization of the paper. In a preliminary Section 2 we describe the appropriate functional setting for the study of problem (P_λ), including the definition of an equivalent problem, with the aid of an extra variable, which provides some advantages, see Remark 2.1. Then we devote Sections 3 and 4 to the proof of Theorems 1.1–1.3. Finally the regularity results, together with a concentration-compactness theorem, are proved in Section 5.

2. Preliminaries and functional setting

The powers $(-\Delta)^{\alpha/2}$ of the positive Laplace operator $(-\Delta)$, in a bounded domain Ω with zero Dirichlet boundary data, are defined through the spectral decomposition using the powers of the eigenvalues of the original operator. Let (φ_j, ρ_j) be the eigenfunctions and eigenvectors of $(-\Delta)$ in Ω with zero Dirichlet boundary data. Then $(\varphi_j, \rho_j^{\alpha/2})$ are the eigenfunctions and eigenvectors of $(-\Delta)^{\alpha/2}$, also with Dirichlet boundary conditions. In fact, the fractional Laplacian $(-\Delta)^{\alpha/2}$ is well defined in the space of functions

$$H_0^{\alpha/2}(\Omega) = \left\{ u = \sum a_j \varphi_j \in L^2(\Omega) \ : \ \|u\|_{H_0^{\alpha/2}(\Omega)} = \left(\sum a_j^2 \rho_j^{\alpha/2} \right)^{1/2} < \infty \right\},$$

and, as a consequence,

$$(-\Delta)^{\alpha/2} u = \sum a_j \rho_j^{\alpha/2} \varphi_j \, .$$

Note that then $\|u\|_{H_0^{\alpha/2}(\Omega)} = \|(-\Delta)^{\alpha/4} u\|_{L^2(\Omega)}$.

The dual space $H^{-\alpha/2}(\Omega)$ is defined in the standard way, as well as the inverse operator $(-\Delta)^{-\alpha/2}$.

We now consider the problem

$$\left\{ \begin{array}{ll} (-\Delta)^{\alpha/2} u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{2.1}$$

in this functional framework. Since the above definition of the fractional Laplacian allows to integrate by parts in the proper spaces, a natural definition of energy solution to problem (2.1) is the following.

Definition 2.1. *We say that $u \in H_0^{\alpha/2}(\Omega)$ is a solution of (2.1) if the identity*

$$\int_\Omega (-\Delta)^{\alpha/4} u (-\Delta)^{\alpha/4} \varphi \, dx = \int_\Omega f(u) \varphi \, dx \tag{2.2}$$

holds for every function $\varphi \in H_0^{\alpha/2}(\Omega)$.

Our problem (P_λ) is like problem (2.1) with $f(u) = f_\lambda(u) = \lambda u^q + u^{\frac{N+\alpha}{N-\alpha}}$. In this case the right-hand side of (2.2) is well defined since $\varphi \in H_0^{\alpha/2}(\Omega) \hookrightarrow L^{\frac{2N}{N-\alpha}}(\Omega)$, while $u \in H_0^{\alpha/2}(\Omega)$ hence $f(u) \in L^{\frac{2N}{N+\alpha}}(\Omega) \hookrightarrow H^{-\alpha/2}(\Omega)$.

Associated to problem (2.1) we consider the energy functional

$$I(u) = \frac{1}{2} \int_\Omega \left| (-\Delta)^{\alpha/4} u \right|^2 dx - \int_\Omega F(u) \, dx \,,$$

where $F(u) = \int_0^u f(s) \, ds$. In our case it reads

$$I(u) = \frac{1}{2} \int_\Omega \left| (-\Delta)^{\alpha/4} u \right|^2 dx - \frac{\lambda}{q+1} \int_\Omega u^{q+1} \, dx - \frac{N-\alpha}{2N} \int_\Omega u^{\frac{2N}{N-\alpha}} \, dx \,. \tag{2.3}$$

This functional is well defined in $H_0^{\alpha/2}(\Omega)$, and moreover, the critical points of I correspond to solutions to (P_λ).

We now include the main ingredients of a recently developed technique used in order to deal with fractional powers of the Laplacian.

Motivated by the work of Caffarelli and Silvestre [13], several authors have considered an equivalent definition of the operator $(-\Delta)^{\alpha/2}$ in a bounded domain with zero Dirichlet boundary data by means of an auxiliary variable, see [5, 11, 12, 14, 22].

Associated to the bounded domain Ω, let us consider the cylinder $\mathcal{C}_\Omega = \Omega \times (0, \infty) \subset \mathbb{R}_+^{N+1}$. The points in $\mathcal{C}_\Omega$ are denoted by (x, y). The lateral boundary of the cylinder will be denoted by $\partial_L \mathcal{C}_\Omega = \partial\Omega \times (0, \infty)$. Now, for a function $u \in H_0^{\alpha/2}(\Omega)$, we define the *$\alpha$-harmonic extension* $w = \mathrm{E}_\alpha(u)$ to the cylinder $\mathcal{C}_\Omega$ as the solution to the problem

$$\begin{cases} \operatorname{div}(y^{1-\alpha} \nabla w) = 0 & \text{in } \mathcal{C}_\Omega, \\ w = 0 & \text{on } \partial_L \mathcal{C}_\Omega, \\ w = u & \text{on } \Omega \times \{y = 0\}. \end{cases} \tag{2.4}$$

The extension function belongs to the space

$$X_0^\alpha(\mathcal{C}_\Omega) = \left\{ z \in L^2(\mathcal{C}_\Omega) \,:\, z = 0 \text{ on } \partial_L \mathcal{C}_\Omega, \ \|z\|_{X_0^\alpha(\mathcal{C}_\Omega)} = \left(\kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha} |\nabla z|^2 \right)^{1/2} < \infty \right\},$$

where κ_α is a normalization constant. With this constant we have that the extension operator is an isometry between $H_0^{\alpha/2}(\Omega)$ and $X_0^\alpha(\mathcal{C}_\Omega)$. That is

$$\| \mathrm{E}_\alpha(\psi) \|_{X_0^\alpha(\mathcal{C}_\Omega)} = \|\psi\|_{H_0^{\alpha/2}(\Omega)} \,, \quad \forall \, \psi \in H_0^{\alpha/2}(\Omega). \tag{2.5}$$

Moreover, for any function $\varphi \in X_0^\alpha(\mathcal{C}_\Omega)$, we have the following trace inequality

$$\|\varphi(\cdot, 0)\|_{H_0^{\alpha/2}(\Omega)} \le \|\varphi\|_{X_0^\alpha(\mathcal{C}_\Omega)}. \tag{2.6}$$

The relevance of the extension function w is that it is related to the fractional Laplacian of the original function u through the formula

$$- \lim_{y \to 0^+} y^{1-\alpha} \frac{\partial w}{\partial y}(x, y) = \frac{1}{\kappa_\alpha} (-\Delta)^{\alpha/2} u(x), \tag{2.7}$$

see [5, 11, 12, 13, 14, 22]. When $\Omega = \mathbb{R}^N$, the above Dirichlet-Neumann procedure (2.4)–(2.7) provides a formula for the fractional Laplacian in the whole space equivalent to that obtained

from Fourier Transform, see [13]. In that case, the α-harmonic extension and the fractional Laplacian have explicit expressions in terms of the Poisson and the Riesz kernels, respectively:

$$
\begin{aligned}
w(x,y) &= P_y^{\alpha} * u(x) = c_{N,\alpha} y^{\alpha} \int_{\mathbb{R}^N} \frac{u(s)}{(|x-s|^2+y^2)^{\frac{N+\alpha}{2}}}\, ds\,, \\
(-\Delta)^{\alpha/2} u(x) &= d_{N,\alpha} P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(s)}{|x-s|^{N+\alpha}}\, ds\,.
\end{aligned} \tag{2.8}
$$

In fact the extension technique is developed originally for the fractional Laplacian defined in the whole space, [13], and the corresponding functional spaces are well defined on the homogeneous fractional Sobolev space $\dot{H}^{\alpha/2}(\mathbb{R}^N)$ and the weighted Sobolev space $X^{\alpha}(\mathbb{R}_+^{N+1})$. The constants in (2.8) and (2.7) satisfy the identity $\alpha c_{N,\alpha}\kappa_\alpha = d_{N,\alpha}$. Their explicit value can be consulted for instance in [5]. We will use the following notation,

$$
L_\alpha w := -\operatorname{div}(y^{1-\alpha}\nabla w), \qquad \frac{\partial w}{\partial \nu^\alpha} := -\kappa_\alpha \lim_{y\to 0^+} y^{1-\alpha}\frac{\partial w}{\partial y}\,.
$$

With this extension, we can reformulate our problem (P_λ) as

$$
(\overline{P}_\lambda) \qquad \begin{cases} L_\alpha w = 0 & \text{in } \mathcal{C}_\Omega \\ w = 0 & \text{on } \partial_L \mathcal{C}_\Omega \\ \dfrac{\partial w}{\partial \nu^\alpha} = \lambda w^q + w^{\frac{N+\alpha}{N-\alpha}} & \text{in } \Omega\times\{y=0\}. \end{cases}
$$

An energy solution to this problem is a function $w \in X_0^\alpha(\mathcal{C}_\Omega)$ such that

$$
\kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}\langle \nabla w, \nabla\varphi\rangle\, dxdy = \int_\Omega \left(\lambda w^q + w^{\frac{N+\alpha}{N-\alpha}}\right)\varphi\, dx, \qquad \forall\, \varphi \in X_0^\alpha(\mathcal{C}_\Omega). \tag{2.9}
$$

For any energy solution $w \in X_0^\alpha(\mathcal{C}_\Omega)$ to this problem, the function $u = w(\cdot,0)$, defined in the sense of traces, belongs to the space $H_0^{\alpha/2}(\Omega)$ and is an energy solution to problem (P_λ). The converse is also true. Therefore, both formulations are equivalent.

The associated energy functional to the problem $(\overline{P}_\lambda)$ is

$$
J(w) = \frac{\kappa_\alpha}{2}\int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2\, dxdy - \frac{\lambda}{q+1}\int_\Omega w^{q+1}\, dx - \frac{N-\alpha}{2N}\int_\Omega w^{\frac{2N}{N-\alpha}}\, dx\,. \tag{2.10}
$$

Clearly, critical points of J in $X_0^\alpha(\mathcal{C}_\Omega)$ correspond to critical points of I in $H_0^{\alpha/2}(\Omega)$. Even more, minima of J also correspond to minima of I, see Section 3.

Remark 2.1. *In the sequel, and in view of the above equivalence, we will use both formulations of the problem, in Ω or in $\mathcal{C}_\Omega$, whenever we may take some advantage. In particular, we will use the extension version $(\overline{P}_\lambda)$ when dealing with the fractional operator acting on products of functions, since it is not clear how to calculate this action. This difficulty appears in the proof of the concentration-compactness result, Theorem 5.1, among others.*

Another tool which is very useful in what follows is the trace inequality

$$
\int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla z(x,y)|^2\, dxdy \geq C\left(\int_\Omega |z(x,0)|^r\, dx\right)^{2/r}, \tag{2.11}
$$

for any $1 \leq r \leq \frac{2N}{N-\alpha}$, $N > \alpha$, and any $z \in X_0^\alpha(\mathcal{C}_\Omega)$, where $C = C(\alpha, r, N, \Omega) > 0$. In fact it is equivalent to the fractional Sobolev inequality

$$
\int_\Omega |(-\Delta)^{\alpha/4} v|^2\, dx \geq C\left(\int_\Omega |v|^r\, dx\right)^{2/r} \tag{2.12}
$$

for any $1 \leq r \leq \frac{2N}{N-\alpha}$, $N > \alpha$, and every $v \in H_0^{\alpha/2}(\Omega)$. In the following we will denote the critical fractional Sobolev exponent $2_\alpha^* = \frac{2N}{N-\alpha}$.

Remark 2.2. *When $r = 2_\alpha^*$, the best constant in* (2.11) *will be denoted by $S(\alpha, N)$. This constant is explicit and independent of the domain; its exact value is*

$$
S(\alpha, N) = \frac{2\pi^{\frac{\alpha}{2}}\Gamma(\frac{N+\alpha}{2})\Gamma(\frac{2-\alpha}{2})(\Gamma(\frac{N}{2}))^{\frac{\alpha}{N}}}{\Gamma(\frac{\alpha}{2})\Gamma(\frac{N-\alpha}{2})(\Gamma(N))^{\frac{\alpha}{2}}}.
$$

It is not achieved in any bounded domain, so we have

$$\int_{\mathbb{R}^{N+1}_+} y^{1-\alpha}|\nabla z(x,y)|^2\,dxdy \geq S(\alpha,N)\left(\int_{\mathbb{R}^N}|z(x,0)|^{\frac{2N}{N-\alpha}}\,dx\right)^{\frac{N-\alpha}{N}}, \quad \forall z\in X^\alpha(\mathbb{R}^{N+1}_+), \tag{2.13}$$

though it is indeed achieved in that case $\Omega=\mathbb{R}^{N+1}_+$ *when* $u=z(\cdot,0)$ *takes the form*

$$u(x)=u_\varepsilon(x)=\frac{\varepsilon^{(N-\alpha)/2}}{(|x|^2+\varepsilon^2)^{(N-\alpha)/2}}, \tag{2.14}$$

with $\varepsilon>0$ *arbitrary and* $z=\mathrm{E}_\alpha(u)$. *See* [5] *for more details. This will be used in Sections 3 and 4. The best constant in* (2.12) *when* $\Omega=\mathbb{R}^N$ *is then* $\kappa_\alpha S(\alpha,N)$.

3. Sublinear case: $0<q<1$.

We prove here Theorem 1.1. As we have said in Remark 2.1, there are some points where it is difficult to work directly with the fractional Laplacian, due to the absence of formula for the fractional Laplacian of a product. Therefore we consider in some occasions the extended problem $(\overline{P}_\lambda)$.

To begin with that problem, we prove that local minima of the functional I correspond to local minima of the extended functional J.

Proposition 3.1. *A function* $u_0\in H_0^{\alpha/2}(\Omega)$ *is a local minimum of* I *if and only if* $w_0=\mathrm{E}_\alpha(u_0)\in X_0^\alpha(\mathcal{C}_\Omega)$ *is a local minimum of* J.

Proof. Firstly let $u_0\in H_0^{\alpha/2}(\Omega)$ be a local minimum of I. Suppose, by contradiction, that $w_0=\mathrm{E}_\alpha(u_0)$ is not a local minimum for the extended functional J. Then by (2.5) and (2.6), we have that, for any $\varepsilon>0$, there exists $w_\varepsilon\in X_0^\alpha(\mathcal{C}_\Omega)$, with $\|w_0-w_\varepsilon\|_{X_0^\alpha(\mathcal{C}_\Omega)}<\varepsilon$, such that

$$I(u_0)=J(w_0)>J(w_\varepsilon)\geq I(z_\varepsilon)$$

where $z_\varepsilon=w_\varepsilon(\cdot,0)\in H_0^{\alpha/2}(\Omega)$ satisfies $\|u_0-z_\varepsilon\|_{H_0^{\alpha/2}(\Omega)}<\varepsilon$.

On the other hand, let $w_0\in X_0^\alpha(\mathcal{C}_\Omega)$ be a local minimum of J. It is clear, from the definition of the extension operator, that w_0 is α-harmonic. So we conclude. □

We return now to the original problem (P_λ), posed at the bottom $\Omega\times\{y=0\}$.

Lemma 3.1. *Let* Λ *be defined by*

$$\Lambda=\sup\{\lambda>0\ :\ \textit{Problem } (P_\lambda) \textit{ has solution}\}.$$

Then $0<\Lambda<\infty$.

Proof. Let (λ_1,φ_1) be the first eigenvalue and a corresponding positive eigenfunction of the fractional Laplacian in Ω. Then, using φ_1 as a test function in (P_λ), we have that

$$\int_\Omega\left(\lambda u^q+u^{\frac{N+\alpha}{N-\alpha}}\right)\varphi_1\,dx=\lambda_1\int_\Omega u\varphi_1\,dx. \tag{3.15}$$

Since there exist positive constants c,δ such that $\lambda t^q+t^{\frac{N+\alpha}{N-\alpha}}>c\lambda^\delta t$, for any $t>0$ we obtain from (3.15) that $c\lambda^\delta<\lambda_1$ which implies $\Lambda<\infty$.

To prove $\Lambda>0$ we use the sub- and supersolution technique to construct a solution for any small λ, see [18, 2]. In fact a subsolution is obtained as a small multiple of φ_1. A supersolution is a large multiple of the function g solution to

$$\begin{cases}(-\Delta)^{\alpha/2}g=1 & \text{in } \Omega,\\ g=0 & \text{on } \partial\Omega.\end{cases}$$

□

Comparison is clear for linear problems associated to the fractional Laplacian, as it is for the Laplacian. On the other hand, it is in general not true for nonlinear problems. Nevertheless, it holds when the reaction term is a nonnegative sublinear function, see [7, 2, 5]. Therefore, it is easy to show, comparing with the problem with only the concave terms λu^q, that in fact there is at least one positive solution u_λ to problem (P_λ) for every λ in the whole interval $(0,\Lambda)$. Even more, these constructed solutions are minimal and are increasing with respect to λ (see Lema 5.7 of [5]).

To prove existence of solution in the extremal value $\lambda = \Lambda$, the idea, like in [2], consists on passing to the limit as $\lambda_n \nearrow \Lambda$ on the sequence $\{z_n\} = \{z_{\lambda_n}\}$, where z_{λ_n} is the minimal solution of $(\overline{P}_\lambda)$ with $\lambda = \lambda_n$. Denote by J_{λ_n} the associated functional. Clearly $J_{\lambda_n}(z_n) < 0$, hence

$$0 > J_{\lambda_n}(z_n) - \frac{1}{2^*_\alpha}\langle J'_{\lambda_n}(z_n), z_n\rangle = \kappa_\alpha\left(\frac{1}{2} - \frac{1}{2^*_\alpha}\right)\|z_n\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} - \lambda_n\left(\frac{1}{q+1} - \frac{1}{2^*_\alpha}\right)\int_\Omega z_n^{q+1}dx.$$

Therefore, by the Sobolev and Trace inequalities, (2.12) and (2.6) respectively, there exits a constant $C > 0$ such that $\|z_n\|_{X_0^\alpha(\mathcal{C}_\Omega)} \le C$. As a consequence, there exists a subsequence weakly convergent to some z_Λ in $X_0^\alpha(\mathcal{C}_\Omega)$. By comparison, $z_\Lambda \ge z_\lambda > 0$, for any $0 < \lambda < \Lambda$, so one gets easily that z_Λ is a weak nontrivial solution to $(\overline{P}_\lambda)$ with $\lambda = \Lambda$.

Having proved the first three items in Theorem 1.1, we focus in the sequel on proving the existence of a second solution, for which we recall that $\alpha \ge 1$.

The proof is divided into several steps: we first show that the minimal solution is a local minimum for the functional I; so we can use the Mountain Pass Theorem, obtaining a minimax Palais-Smale (PS) sequence. In the next step, in order to find a second solution, we prove a local $(\mathrm{PS})_c$ condition for c under a critical level c^*. To do that, we will construct path by localizing the minimizers of the Trace/Sobolev inequalities at the possible Dirac Deltas, given by the concentration-compactness result in Theorem 5.1.

We begin with a separation lemma in the $\mathcal{C}^1$-topology.

Lemma 3.2. *Let $0 < \mu_1 < \lambda_0 < \mu_2 < \Lambda$. Let z_{μ_1}, z_{λ_0} and z_{μ_2} be the corresponding minimal solutions to (P_λ), $\lambda = \mu_1$, λ_0 and μ_2 respectively. If $X = \{z \in \mathcal{C}_0^1(\Omega) \,|\, z_{\mu_1} \le z \le z_{\mu_2}\}$, then there exists $\varepsilon > 0$ such that*

$$\{z_{\lambda_0}\} + \varepsilon B_1 \subset X,$$

where B_1 is the unit ball in $\mathcal{C}_0^1(\Omega)$.

Proof. Since $\alpha \ge 1$, we have that any solution u to (P_λ), for arbitrary $0 < \lambda < \Lambda$ belongs to $\mathcal{C}^{1,\gamma}(\overline{\Omega})$ for some positive γ, see Proposition 5.2. Therefore, we deduce that there exists a positive constant C such that

$$u(x) \le C\,\mathrm{dist}(x, \partial\Omega),\ x \in \Omega. \tag{3.16}$$

On the other hand, by comparison with the first eigenfunction of the fractional Laplacian (which is indeed the first eigenfunction φ_1 of the classical Laplacian), we get that there exists a positive constant c such that

$$u(x) \ge c\,\mathrm{dist}(x, \partial\Omega),\ x \in \Omega. \tag{3.17}$$

These two estimates jointly with the regularity implies the result of the lemma. $\square$

With this result we now obtain a local minimum of the functional I in $\mathcal{C}_0^1(\Omega)$, as a first step, to obtain a local minimum in $H_0^{\alpha/2}(\Omega)$.

Lemma 3.3. *For all $\lambda \in (0, \Lambda)$ there exists a solution for (P_λ) which is a local minimum of the functional I in the $\mathcal{C}^1$-topology.*

Proof. Given $0 < \mu_1 < \lambda < \mu_2 < \Lambda$, let z_{μ_1} and z_{μ_2} be the minimal solutions of (P_{μ_1}) and (P_{μ_2}) respectively. Let $z := z_{\mu_2} - z_{\mu_1}$. Since z_{μ_1} and z_{μ_2} are properly ordered, then

$$\begin{cases} (-\Delta)^{\alpha/2} z \ge 0 & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases}$$

We set

$$f^*(x, s) = \begin{cases} f_\lambda(z_{\mu_1}(x)) & \text{if } s \le z_{\mu_1}, \\ f_\lambda(s) & \text{if } z_{\mu_1} \le s \le z_{\mu_2}, \\ f_\lambda(z_{\mu_2}(x)) & \text{if } z_{\mu_2} \le s, \end{cases}$$

$$F^*(x, z) = \int_0^z f^*(x, s)\, ds$$

and

$$I^*(z) = \frac{1}{2}\|z\|_{H_0^{\alpha/2}(\Omega)} - \int_\Omega F^*(x, u)dx.$$

Standard calculation shows that I^* achieves its global minimum at some $u_0 \in H_0^{\alpha/2}(\Omega)$, that is

$$I^*(u_0) \le I^*(z) \quad \forall\, z \in H_0^{\alpha/2}(\Omega). \tag{3.18}$$

Moreover it holds

$$\begin{cases} (-\Delta)^{\alpha/2} u_0 = f^*(x, u_0) & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

By Lemma 3.2, it follows that $\{u_0\} + \varepsilon B_1 \subseteq X$ for $0 < \varepsilon$ small enough. Let now z satisfying

$$\|z - u_0\|_{\mathcal{C}_0^1(\Omega)} \le \frac{\varepsilon}{2}.$$

As $I^*(z) - I(z)$ is zero for every z such that $\|z - u_0\|_{\mathcal{C}_0^1(\Omega)} \le \frac{\varepsilon}{2}$, by (3.18) we obtain that

$$I(z) = I^*(z) \ge I^*(u_0) = I(u_0), \quad \forall\, z \in \mathcal{C}_0^1(\Omega), \text{ with } \|z - u_0\|_{\mathcal{C}_0^1(\Omega)} \le \frac{\varepsilon}{2}.$$

□

To show that we have obtained the desired minimum in $H_0^{\alpha/2}(\Omega)$, we now check that the result by Brezis and Nirenberg in [10] is also valid in our context.

Proposition 3.2. *Let $z_0 \in H_0^{\alpha/2}(\Omega)$ be a local minimum of I in $\mathcal{C}_0^1(\Omega)$, i.e., there exists $r > 0$ such that*

$$I(z_0) \le I(z_0 + z) \qquad \forall z \in \mathcal{C}_0^1(\Omega) \text{ with } \|z\|_{\mathcal{C}_0^1(\Omega)} \le r. \tag{3.19}$$

Then z_0 is a local minimum of I in $H_0^{\alpha/2}(\Omega)$, that is, there exists $\varepsilon_0 > 0$ such that

$$I(z_0) \le I(z_0 + z) \qquad \forall z \in H_0^{\alpha/2}(\Omega) \text{ with } \|z\|_{H_0^{\alpha/2}(\Omega)} \le \varepsilon_0.$$

Proof. Arguing by contradiction we suppose that

$$\forall\, \varepsilon > 0,\, \exists\, z_\varepsilon \in B_\varepsilon(z_0) \quad \text{such that} \quad I(z_\varepsilon) < I(z_0),$$

where $B_\varepsilon(z_0) = \left\{ z \in H_0^{\alpha/2}(\Omega) : \ \|z - z_0\|_{H_0^{\alpha/2}(\Omega)} \le \varepsilon \right\}$.

For every $j > 0$ we consider the truncation map given by

$$T_j(r) \equiv \begin{cases} r & 0 < r < j, \\ j & r \ge j. \end{cases}$$

Let

$$f_{\lambda,j}(s) = f_\lambda(T_j(s)), \qquad F_j(s) = \int_0^u f_{\lambda,j}(s) ds\,,\; u > 0\,,$$

and

$$I_j(z) = \frac{1}{2}\|z\|^2_{H_0^{\alpha/2}(\Omega)} - \int_\Omega F_j(z) dx.$$

Note that for each $z \in H_0^{\alpha/2}(\Omega)$ we have that $I_j(z) \to I(z)$ as $j \to \infty$. Hence, for each $\varepsilon > 0$ there exists $j(\varepsilon)$ big enough such that $I_{j(\varepsilon)}(z_\varepsilon) < I(z_0)$. Clearly $\min\limits_{B_\varepsilon(z_0)} I_{j(\varepsilon)}$ is attained at some point, say v_ε. Thus we have

$$I_{j(\varepsilon)}(v_\varepsilon) \le I_{j(\varepsilon)}(z_\varepsilon) < I(z_0).$$

Now we want to prove that $v_\varepsilon \to z_0$ in $\mathcal{C}_0^1(\Omega)$ as $\varepsilon \searrow 0$. The Euler-Lagrange equation satisfied by v_ε involves a Lagrange multiplier ξ_ε in such a way that

$$\langle I'_{j(\varepsilon)}(v_\varepsilon), \varphi \rangle_{H^{-\alpha/2}(\Omega), H_0^{\alpha/2}(\Omega)} = \xi_\varepsilon \langle v_\varepsilon, \varphi \rangle_{H_0^{\alpha/2}(\Omega)}, \quad \forall\, \varphi \in H_0^{\alpha/2}(\Omega). \tag{3.20}$$

Since v_ε is a minimum of $I_{j(\varepsilon)}$, it holds

$$\xi_\varepsilon = \frac{\langle I'_{j(\varepsilon)}(v_\varepsilon), v_\varepsilon \rangle}{\|v_\varepsilon\|^2_{H_0^{\alpha/2}(\Omega)}} \le 0 \quad \text{for } 0 < \varepsilon \ll 1, \quad \text{and} \quad \xi_\varepsilon \to 0 \text{ as } \varepsilon \searrow 0. \tag{3.21}$$

Note that by (3.20), v_ε satisfies the problem

$$\begin{cases} (-\Delta)^{\alpha/2} v_\varepsilon = \frac{1}{1-\xi_\varepsilon} f_{\lambda,j(\varepsilon)}(v_\varepsilon) := f^\varepsilon_{\lambda,j(\varepsilon)}(v_\varepsilon) & \text{in } \Omega, \\ v_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Clearly $\|v_\varepsilon\|_{H_0^{\alpha/2}(\Omega)} \le C$, thus, by Proposition 5.1, this implies that $\|v_\varepsilon\|_{L^\infty(\Omega)} \le C$. Moreover, by (3.21) it follows that $\|f^\varepsilon_{\lambda,j(\varepsilon)}(v_\varepsilon)\|_{L^\infty(\Omega)} \le C$. Therefore, following the proof of Proposition

5.2, we get that $\|v_\varepsilon\|_{\mathcal{C}^{1,r}(\overline{\Omega})} \leq C$, for $r = \min\{q, \alpha-1\}$ and C independent of ε. By Ascoli-Arzelá Theorem there exists a subsequence, still denoted by v_ε, such that $v_\varepsilon \to z_0$ uniformly in $\mathcal{C}^1_0(\Omega)$ as $\varepsilon \searrow 0$. This implies that for ε small enough,

$$I(v_\varepsilon) = I_{j(\varepsilon)}(v_\varepsilon) < I(z_0)$$

for any v_ε with $\|v_\varepsilon - z_0\|_{\mathcal{C}^1_0(\Omega)} < \varepsilon$. □

Lemma 3.3 and Proposition 3.2 provide us a local minimum in $H^{\alpha/2}_0(\Omega)$, which will be denoted by u_0. We now perform a traslation in order to simplify the calculations.

We consider the functions

$$g(x,s) = \begin{cases} \lambda(u_0+s)^q - \lambda u_0^q + (u_0+s)^{2^*_\alpha - 1} - u_0^{2^*_\alpha - 1} & \text{if } s \geq 0, \\ 0 & \text{if } s < 0, \end{cases} \tag{3.22}$$

$$G(u) = \int_0^u g(x,s)\,ds, \tag{3.23}$$

and the energy functional

$$\widetilde{I}(u) = \frac{1}{2}\|u\|^2_{H^{\alpha/2}_0(\Omega)} - \int_\Omega G(x,u)dx. \tag{3.24}$$

Since $u \in H^{\alpha/2}_0(\Omega)$, G is well defined and bounded from below. Let the moved problem

$$(\widetilde{P}_\lambda) \quad \begin{cases} (-\Delta)^{\alpha/2} u = g(x,u) & \text{in } \Omega \subset \mathbb{R}^N,\ \lambda > 0 \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, by standard variational theory, we know that if $\widetilde{u} \not\equiv 0$ is a critical point of $\widetilde{I}$ then it is a solution of $(\widetilde{P}_\lambda)$ which, by the Maximum Principle (Lemma 2.3 of [14]), it is $\widetilde{u} > 0$. Therefore $u = u_0 + \widetilde{u}$ will be a second solution of (P_λ) for the sublinear case. Thus we will need to study the existence of these non-trivial critical points for $\widetilde{I}$.

Firstly we have

Lemma 3.4. *$u = 0$ is a local minimum of $\widetilde{I}$ in $H^{\alpha/2}_0(\Omega)$.*

Proof. The proof follows the lines of [2], so we will be brief in details. Note that by Proposition 3.2 it is sufficient to prove that $u = 0$ is a local minimum of $\widetilde{I}$ in $\mathcal{C}^1_0(\Omega)$.

Let $u \in \mathcal{C}^1_0(\Omega)$, then

$$G(u) = F(u_0 + u) - F(u_0) - \left(\lambda u_0^q + u_0^{2^*_\alpha - 1}\right) u. \tag{3.25}$$

Therefore

$$\begin{aligned} \widetilde{I}(u) &= \frac{1}{2}\|u\|^2_{H^{\alpha/2}_0(\Omega)} - \int_\Omega G(u)dx \\ &= \frac{1}{2}\|u\|^2_{H^{\alpha/2}_0(\Omega)} - \int_\Omega F(u_0+u)dx + \int_\Omega F(u_0)dx + \int_\Omega \left(\lambda u_0^q + u_0^{2^*_\alpha - 1}\right) u dx. \end{aligned}$$

On the other hand,

$$\begin{aligned} I(u_0+u) &= \frac{1}{2}\|u_0+u\|^2_{H^{\alpha/2}_0(\Omega)} - \int_\Omega F(u_0+u)dx \\ &= \frac{1}{2}\|u_0\|^2_{H^{\alpha/2}_0(\Omega)} + \frac{1}{2}\|u\|^2_{H^{\alpha/2}_0(\Omega)} + \int_\Omega (-\Delta)^{\alpha/4}u_0(-\Delta)^{\alpha/4}u dx - \int_\Omega F(u_0+u)dx \\ &= \frac{1}{2}\|u_0\|^2_{H^{\alpha/2}_0(\Omega)} + \frac{1}{2}\|u\|^2_{H^{\alpha/2}_0(\Omega)} + \int_\Omega \left(\lambda u_0^q + u_0^{2^*_\alpha - 1}\right) u dx - \int_\Omega F(u_0+u)dx. \end{aligned}$$

Finally, as u_0 is a local minimum of I, we have that

$$\begin{aligned} \widetilde{I}(u) &= I(u_0+u) - \frac{1}{2}\|u_0\|^2_{H^{\alpha/2}_0(\Omega)} + \int_\Omega F(u_0)dx \\ &= I(u_0+u) - I(u_0) \\ &\geq 0 = \widetilde{I}(0) \end{aligned}$$

provided $\|u\|_{\mathcal{C}^1_0(\Omega)} < \varepsilon$. □

As a consequence of Proposition 3.1, we obtain for the moved functional

$$\widetilde{J}(w) = \frac{1}{2}\|w\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} - \int_\Omega G(w(x,0))dx,$$

with G as in (3.22)-(3.23), the following result.

Corollary 3.1. *$w = 0$ is a local minimum of $\widetilde{J}$ in $X_0^\alpha(\mathcal{C}_\Omega)$.*

Now assuming that $v = 0$ is the unique critical point of the moved functional $\widetilde{J}$, then a local $(\mathrm{PS})_c$ condition can be proved for c under a critical level c^*,

$$c^* = \frac{\alpha}{2N}(\kappa_\alpha S(\alpha, N))^{\frac{N}{\alpha}}. \tag{3.26}$$

Following the ideas given in [2], and by an extension of a concentration-compactness result by Lions, that we prove in Theorem 5.1, we obtain the following result.

Lemma 3.5. *If $v = 0$ is the only critical point of $\widetilde{J}$ in $X_0^\alpha(\mathcal{C}_\Omega)$ then $\widetilde{J}$ satisfies a local Palais Smale condition below the critical level c^*.*

Proof. Let $\{w_n\}$ be a Palais-Smale sequence for $\widetilde{J}$ verifying

$$\widetilde{J}(w_n) \to c < c^*, \qquad \widetilde{J}'(w_n) \to 0. \tag{3.27}$$

Since the fact that w_0 is a critical point implies $\widetilde{J}(w_n) = J(z_n) - J(w_0)$, where $z_n = w_n + w_0$, we have that

$$J(z_n) \to c + J(w_0), \qquad J'(z_n) \to 0. \tag{3.28}$$

On the other hand, from (3.27) we get that the sequence $\{z_n\}$ is uniformly bounded in $X_0^\alpha(\mathcal{C}_\Omega)$. As a consequence, up to a subsequence,

$$\begin{array}{rclll} z_n & \rightharpoonup & z & \text{weakly in } X_0^\alpha(\mathcal{C}_\Omega) & \\ z_n(\cdot,0) & \to & z(\cdot,0) & \text{strong in } L^r(\Omega), & \forall\, 1 \le r < 2^*_\alpha \\ z_n(\cdot,0) & \to & z(\cdot,0) & \text{a.e. in } \Omega. & \end{array} \tag{3.29}$$

Note that as $v = 0$ is the unique critical point of $\widetilde{J}$ then, $z = w_0$.

In order to apply the concentration-compactness result, Theorem 5.1, first we prove the following.

Lemma 3.6. *The sequence $\left\{y^{1-\alpha}|\nabla z_n|^2\right\}_{n\in\mathbb{N}}$ is tight, i.e., for any $\eta > 0$ there exists $\rho_0 > 0$ such that*

$$\int_{\{y>\rho_0\}}\int_\Omega y^{1-\alpha}|\nabla z_n|^2 dxdx \le \eta, \quad \forall\, n \in \mathbb{N}. \tag{3.30}$$

Proof. The proof of this lemma follows some arguments of Lema 2.2 in [4]. By contradiction, we suppose that there exits $\eta_0 > 0$ such that, for any $\rho > 0$ one has, up to a subsequence,

$$\int_{\{y>\rho\}}\int_\Omega y^{1-\alpha}|\nabla z_n|^2 dxdy > \eta_0 \quad \text{for every } n \in \mathbb{N}. \tag{3.31}$$

Let $\varepsilon > 0$ be fixed (to be precised later), and let $r > 0$ be such that

$$\int_{\{y>r\}}\int_\Omega y^{1-\alpha}|\nabla z|^2 dxdy < \varepsilon.$$

Let $j = \left[\frac{M}{\kappa_\alpha \varepsilon}\right]$ be the integer part and $I_k = \{y \in \mathbb{R}^+ :\ r + k \le y \le r + k + 1\}$, $k = 0, 1, \ldots, j$. Since $\|z_n\|_{X_0^\alpha(\mathcal{C}_\Omega)} \le M$, we clearly obtain that

$$\sum_{k=0}^{j}\int_{I_k}\int_\Omega y^{1-\alpha}|\nabla z_n|^2 dxdy \le \int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla z_n|^2 dxdy \le \varepsilon(j+1).$$

Therefore there exists $k_0 \in \{0, \ldots, j\}$ such that (again up to a subsequence)

$$\int_{I_{k_0}}\int_\Omega y^{1-\alpha}|\nabla z_n|^2 dxdy \le \varepsilon, \quad \forall\, n. \tag{3.32}$$

Let $\chi \geq 0$ be the following regular non-decreasing cut-off function

$$\chi(y) = \begin{cases} 0 & \text{if } y \leq r + k_0, \\ 1 & \text{if } y > r + k_0 + 1, \end{cases}$$

Define $v_n(x,y) = \chi(y) z_n(x,y)$. Since $v_n(x,0) = 0$ it follows that

$$\begin{aligned} |\langle J'(z_n) - J'(v_n), v_n\rangle| &= \kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha} \langle \nabla(z_n - v_n), \nabla v_n\rangle dxdy \\ &= \kappa_\alpha \int_{I_{k_0}} \int_\Omega y^{1-\alpha} \langle \nabla(z_n - v_n), \nabla v_n\rangle dxdy. \end{aligned}$$

Moreover by the Cauchy-Schwartz inequality, (3.32) and the compact inclusion $H^1(I_{k_0} \times \Omega, y^{1-\alpha})$ into $L^2(I_{k_0} \times \Omega, y^{1-\alpha})$, we have

$$\begin{aligned} |\langle J'(z_n) - J'(v_n), v_n\rangle| &\leq \kappa_\alpha \left(\int_{I_{k_0}} \int_\Omega y^{1-\alpha} |\nabla(z_n - v_n)|^2 dxdy \right)^{\frac{1}{2}} \left(\int_{I_{k_0}} \int_\Omega y^{1-\alpha} |\nabla v_n|^2 dxdy \right)^{\frac{1}{2}} \\ &\leq C\, \kappa_\alpha\, \varepsilon. \end{aligned}$$

On the other hand, by (3.28), we get

$$|\langle J'(v_n), v_n\rangle| \leq C\, \kappa_\alpha\, \varepsilon + o(1).$$

So, for n sufficiently large,

$$\int_{\{y > r+k_0+1\}} \int_\Omega y^{1-\alpha} |\nabla z_n|^2 dxdy \leq \int_{\mathcal{C}_\Omega} y^{1-\alpha} |\nabla v_n|^2 dxdy = \frac{\langle J'(v_n), v_n\rangle}{\kappa_\alpha} \leq C\, \varepsilon.$$

This is a contradiction with (3.31), which proves Lemma 3.6. □

Proof of Lemma 3.5 (cont.). In view of the previous result we can apply Theorem 5.1. Therefore, up to a subsequence, there exists an index set I, at most countable, a sequence of points $\{x_k\} \subset \Omega$, and nonnegative real numbers μ_k, ν_k, such that

$$y^{1-\alpha} |\nabla z_n|^2 \to \mu \geq y^{1-\alpha} |\nabla w_0|^2 + \sum_{k \in I} \mu_k \delta_{x_k} \tag{3.33}$$

and

$$|z_n(\cdot, 0)|^{2^*_\alpha} \to \nu = |w_0(\cdot, 0)|^{2^*_\alpha} + \sum_{k \in I} \nu_k \delta_{x_k} \tag{3.34}$$

in the sense of measures, satisfying also the relation $\mu_k \geq S(\alpha, N) \nu_k^{\frac{2}{2^*_\alpha}}$, for every $k \in I$.

We fix any $k_0 \in I$, and let $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^{N+1}_+)$ be a nonincreasing cut-off function verifying $\phi = 1$ in $B_1^+(x_{k_0})$, $\phi = 0$ in $B_2^+(x_{k_0})^c$. Let now $\phi_\varepsilon(x,y) = \phi(x/\varepsilon, y/\varepsilon)$, clearly $|\nabla \phi_\varepsilon| \leq \frac{C}{\varepsilon}$. We denote $\Gamma_{2\varepsilon} = B_{2\varepsilon}^+(x_{k_0}) \cap \{y = 0\}$. Then, using $\phi_\varepsilon z_n$ as a test function in (3.28), we have

$$\begin{aligned} &\kappa_\alpha \lim_{n\to\infty} \int_{\mathcal{C}_\Omega} y^{1-\alpha} \langle \nabla z_n, \nabla \phi_\varepsilon\rangle z_n dxdy \\ &= \lim_{n\to\infty} \left(\int_{\Gamma_{2\varepsilon}} |z_n|^{2^*_\alpha} \phi_\varepsilon\, dx + \lambda \int_{\Gamma_{2\varepsilon}} |z_n|^{q+1} \phi_\varepsilon\, dx - \kappa_\alpha \int_{B_{2\varepsilon}^+(x_{k_0})} y^{1-\alpha} |\nabla z_n|^2 \phi_\varepsilon\, dxdy \right). \end{aligned}$$

By (3.29), (3.33) and (3.34) we get

$$\begin{aligned} &\lim_{n\to\infty} \kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha} \langle \nabla z_n, \nabla \phi_\varepsilon\rangle z_n\, dxdy \\ &= \int_{\Gamma_{2\varepsilon}} \phi_\varepsilon\, d\nu + \lambda \int_{\Gamma_{2\varepsilon}} |w_0|^{q+1} \phi_\varepsilon\, dx - \kappa_\alpha \int_{B_{2\varepsilon}^+(x_{k_0})} \phi_\varepsilon\, d\mu. \end{aligned} \tag{3.35}$$

On the other hand, using Theorem 1.6 in [17], with $w = y^{1-\alpha} \in A_2$ and $k = 1$, we obtain that

$$\begin{aligned} \left(\int_{B_{2\varepsilon}^+(x_{k_0})} y^{1-\alpha} |\nabla \phi_\varepsilon|^2 |z_n|^2 dxdy \right)^{1/2} &\leq \frac{2}{\varepsilon} \left(\int_{B_{2\varepsilon}^+(x_{k_0})} y^{1-\alpha} |z_n|^2 dxdy \right)^{1/2} \\ &\leq C \left(\int_{B_{2\varepsilon}^+(x_{k_0})} y^{1-\alpha} |\nabla z_n|^2 dxdy \right)^{1/2}. \end{aligned}$$

Since $z_n \in X_0^\alpha(\mathcal{C}_\Omega)$, the last expression goes to zero as $\varepsilon \to 0$. Therefore

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} \left| \int_{\mathcal{C}_\Omega} y^{1-\alpha} \langle \nabla z_n, \nabla \phi_\varepsilon \rangle z_n dxdy \right| \\ &\leq \lim_{n\to\infty} \left(\int_{\mathcal{C}_\Omega} y^{1-\alpha} |\nabla z_n|^2 dxdy \right)^{1/2} \left(\int_{B_{2\varepsilon}^+(x_{k_0})} y^{1-\alpha} |\nabla \phi_\varepsilon|^2 |z_n|^2 dxdy \right)^{1/2} \longrightarrow 0. \end{aligned}$$

Hence, by (3.35), it follows that

$$\lim_{\varepsilon\to 0} \left[\int_{\Gamma_{2\varepsilon}} \phi_\varepsilon \, d\nu + \lambda \int_{\Gamma_{2\varepsilon}} |w_0|^{q+1} \phi_\varepsilon \, dx - \kappa_\alpha \int_{B_{2\varepsilon}^+(x_{k_0})} \phi_\varepsilon \, d\mu \right] = \nu_{k_0} - \kappa_\alpha \mu_{k_0} = 0.$$

Therefore we get that

$$\nu_{k_0} = 0 \qquad \text{or} \qquad \nu_{k_0} \geq (\kappa_\alpha S(\alpha, N))^{\frac{N}{\alpha}}.$$

Suppose that $\nu_{k_0} \neq 0$. It follows that

$$\begin{aligned} c + J(w_0) &= \lim_{n\to\infty} J(z_n) - \frac{1}{2} \langle J'(z_n), z_n \rangle \\ &\geq \frac{\alpha}{2N} \int_\Omega w_0^{2_\alpha^*} dx + \frac{\alpha}{2N} \nu_{k_0} + \lambda \left(\frac{1}{2} - \frac{1}{q+1} \right) \int_\Omega w_0^{q+1} dx \\ &\geq J(w_0) + \frac{\alpha}{2N} (\kappa_\alpha S(\alpha, N))^{N/\alpha} = J(w_0) + c^*. \end{aligned}$$

Then we get a contradiction with (3.27), and since k_0 was arbitrary, $\nu_k = 0$ for all $k \in I$. Hence as a consequence, $u_n \to u_0$ in $L^{2_\alpha^*}(\Omega)$. We finish in the standard way: convergence of u_n in $L^{\frac{2N}{N-\alpha}}(\Omega)$ implies convergence of $f(u_n)$ in $L^{\frac{2N}{N+\alpha}}(\Omega)$, and finally by using the continuity of the inverse operator $(-\Delta)^{-\alpha/2}$, we obtain convergence of u_n in $H_0^{\alpha/2}(\Omega)$. □

Now it remains to show that we can obtain a local $(\mathrm{PS})_c$ sequence for $\widetilde{J}$ under the critical level $c = c^*$. To do that we will use $w_\varepsilon = \mathrm{E}_\alpha(u_\varepsilon)$, the family of minimizers to the Trace inequality (2.13), where u_ε is given in (2.14). We remark that, despite the cases $\alpha = 1$ and $\alpha = 2$, w_ε does not possesses an explicit expression. This is an extra difficulty that we have to overcome. Taking into account that the family u_ε is self-similar, $u_\varepsilon(x) = \varepsilon^{\frac{\alpha-N}{2}} u_1(x/\varepsilon)$ and the fact that the Poisson kernel (2.8) is also self-similar

$$P_y^\alpha(x) = \frac{1}{y^N} P_1^\alpha \left(\frac{x}{y} \right), \tag{3.36}$$

gives easily that the family w_ε satisfies

$$w_\varepsilon(x, y) = \varepsilon^{\frac{\alpha-N}{2}} w_1 \left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon} \right). \tag{3.37}$$

We will denote $P^\alpha = P_1^\alpha$. Also, we will write $w_{1,\alpha}$ instead of w_1 to emphasize the dependence on the parameter α.

Lemma 3.7. *With the above notation it holds*

$$|\nabla w_{1,\alpha}(x, y)| \leq \frac{C}{y} w_{1,\alpha}(x, y), \quad \alpha > 0, \ (x, y) \in \mathbb{R}_+^{N+1} \tag{3.38}$$

and

$$|\nabla w_{1,\alpha}(x, y)| \leq C w_{1,\alpha-1}(x, y), \quad \alpha > 1, \ (x, y) \in \mathbb{R}_+^{N+1}. \tag{3.39}$$

Proof. Differentiating with respect to each variable x_i, $i = 1, \ldots, N$, and the variable y, it follows that

$$\begin{aligned} |\partial_{x_i} w_{1,\alpha}(x, y)| &\leq \int_{\mathbb{R}^N} \frac{(N+\alpha) y^\alpha |x - z|}{(y^2 + |x-z|^2)^{\frac{N+\alpha}{2}+1} (1 + |z|^2)^{\frac{N-\alpha}{2}}} dz \\ &\leq \frac{N+\alpha}{2y} \int_{\mathbb{R}^N} \frac{y^\alpha}{(y^2 + |x-z|^2)^{\frac{N+\alpha}{2}} (1 + |z|^2)^{\frac{N-\alpha}{2}}} dz \\ &= \frac{C}{y} w_{1,\alpha}(x, y) \end{aligned}$$

and

$$\begin{aligned}
|\partial_y w_{1,\alpha}(x,y)| &= \left| \int_{\mathbb{R}^N} \frac{y^{\alpha-1}(\alpha|x-z|^2 - Ny^2)}{(y^2+|x-z|^2)^{\frac{N+\alpha}{2}+1}(1+|z|^2)^{\frac{N-\alpha}{2}}} dz \right| \\
&\leq C \int_{\mathbb{R}^N} \frac{y^{\alpha-1}}{(y^2+|x-z|^2)^{\frac{N+\alpha}{2}}(1+|z|^2)^{\frac{N-\alpha}{2}}} dz \\
&= \frac{C}{y} w_{1,\alpha}(x,y).
\end{aligned}$$

Therefore we get (3.38). To obtain (3.39) we recall that $u_{1,\alpha}(z) = (1+|z|^2)^{-\frac{N-\alpha}{2}}$. Then, by (3.36) it follows that

$$\begin{aligned}
|\partial_y w_{1,\alpha}(x,y)| &= \left| \partial_y \left(\int_{\mathbb{R}^N} \frac{1}{y^N} P^\alpha\left(\frac{x-z}{y}\right) u_{1,\alpha}(z) dz \right) \right| \\
&= \left| -\partial_y \left(\int_{\mathbb{R}^N} P^\alpha(\tilde{z}) u_{1,\alpha}(x - y\tilde{z}) d\tilde{z} \right) \right| \\
&= \left| \int_{\mathbb{R}^N} P^\alpha(\tilde{z}) \langle \tilde{z}, \nabla u_{1,\alpha}(x-y\tilde{z}) \rangle d\tilde{z} \right| \\
&= \left| - \int_{\mathbb{R}^N} \frac{1}{y^N} P^\alpha\left(\frac{x-z}{y}\right) \langle \frac{x-z}{y}, \nabla u_{1,\alpha}(z) \rangle dz \right| \\
&\leq (N-\alpha) \int_{\mathbb{R}^N} \frac{1}{y^N} P^\alpha\left(\frac{x-z}{y}\right) \frac{|x-z|}{y} \frac{|z|}{(1+|z|^2)^{\frac{N-\alpha}{2}+1}} dz \\
&\leq (N-\alpha) \int_{\mathbb{R}^N} \frac{y^{\alpha-1}}{(y^2+|x-z|^2)^{\frac{N+\alpha-1}{2}}(1+|z|^2)^{\frac{N-\alpha+1}{2}}} dz \\
&= C w_{1,\alpha-1}(x,y).
\end{aligned}$$

Doing the same calculations in variables x_i for $i = 1, \ldots, N$, we obtain

$$\begin{aligned}
|\partial_{x_i} w_{1,\alpha}(x,y)| &= \left| -\partial_{x_i} \left(\int_{\mathbb{R}^N} P^\alpha(\tilde{z}) u_{1,\alpha}(x-y\tilde{z}) d\tilde{z} \right) \right| \\
&\leq \int_{\mathbb{R}^N} P^\alpha(\tilde{z}) |\nabla u_{1,\alpha}|(x-y\tilde{z}) d\tilde{z} \\
&= \int_{\mathbb{R}^N} \frac{1}{y^N} P^\alpha\left(\frac{x-z}{y}\right) |\nabla u_{1,\alpha}|(z) dz \\
&\leq (N-\alpha) \int_{\mathbb{R}^N} \frac{y^\alpha}{(y^2+|x-z|^2)^{\frac{N+\alpha}{2}}} \frac{|z|}{(1+|z|^2)^{\frac{N-\alpha}{2}+1}} dz \\
&= C w_{1,\alpha-1}(x,y).
\end{aligned}$$

□

Let us now introduce a cut-off function $\phi_0(s) \in C^\infty(\mathbb{R}_+)$, nonincreasing satisfying

$$\phi_0(s) = 1 \text{ if } 0 \leq s \leq \frac{1}{2}, \quad \phi_0(s) = 0 \text{ if } s \geq 1.$$

Assume without loss of generality that $0 \in \Omega$. We then define, for some fixed $r > 0$ small enough such that $\overline{B}_r^+ \subseteq \overline{\mathcal{C}}_\Omega$, the function $\phi(x,y) = \phi_r(x,y) = \phi_0(\frac{r_{xy}}{r})$ with $r_{xy} = |(x,y)| = (|x|^2+y^2)^{1/2}$. Note that $\phi\omega_\varepsilon \in X_0^\alpha(\mathcal{C}_\Omega)$. Thus we get

Lemma 3.8. *With the above notation, the family $\{\phi w_\varepsilon\}$, and its trace on $\{y = 0\}$, namely $\{\phi u_\varepsilon\}$, satisfy*

$$\|\phi w_\varepsilon\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} = \|w_\varepsilon\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} + O(\varepsilon^{N-\alpha}), \tag{3.40}$$

$$\|\phi u_\varepsilon\|^2_{L^2(\Omega)} = \begin{cases} C\varepsilon^\alpha + O(\varepsilon^{N-\alpha}) & \text{if } N > 2\alpha, \\ C\varepsilon^\alpha \log(1/\varepsilon) + O(\varepsilon^\alpha) & \text{if } N = 2\alpha, \end{cases} \tag{3.41}$$

and

$$\|\phi u_\varepsilon\|^r_{L^r(\Omega)} \geq c\varepsilon^{\frac{N-\alpha}{2}}, \quad \alpha < N < 2\alpha, \quad r = \frac{N+\alpha}{N-\alpha}, \tag{3.42}$$

for ε small enough and $C > 0$.

Proof. The product ϕw_ε satisfies

$$\begin{aligned}\|\phi w_\varepsilon\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} &= \kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}(|\phi\nabla w_\varepsilon|^2 + |w_\varepsilon\nabla\phi|^2 + 2\langle w_\varepsilon\nabla\phi, \phi\nabla w_\varepsilon\rangle)dxdy \\ &\le \|w_\varepsilon\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} + \kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}|w_\varepsilon\nabla\phi|^2 dxdy \\ &\quad + 2\kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}\langle w_\varepsilon\nabla\phi, \phi\nabla w_\varepsilon\rangle dxdy.\end{aligned} \tag{3.43}$$

To estimate the second term of the right hand side, we observe that $0 \le u_\varepsilon(x) \le \varepsilon^{\frac{N-\alpha}{2}}|x|^{\alpha-N}$, and since the extension of the function $\Gamma(x) = |x|^{\alpha-N}$ is $\widetilde{\Gamma}(x,y) = (|x|^2+y^2)^{\frac{\alpha-N}{2}} = r_{xy}^{\alpha-N}$, we get that

$$\begin{aligned}\int_{\mathcal{C}_\Omega} y^{1-\alpha}|w_\varepsilon\nabla\phi|^2 dxdy &\le C\int_{\{\frac{r}{2}\le r_{xy}\le r\}} y^{1-\alpha}w_\varepsilon^2 dxdy \\ &\le C\varepsilon^{N-\alpha}\int_{\{\frac{r}{2}\le r_{xy}\le r\}} y^{1-\alpha} r_{xy}^{2(\alpha-N)} dxdy \\ &= O(\varepsilon^{N-\alpha}).\end{aligned} \tag{3.44}$$

For the remaining term we need to use the properties of the function w_ε given in Proposition 3.7. Let $C_r = \{r/2 \le r_{xy} \le r\} \subset \mathcal{C}_\Omega$. By (3.37) we get

$$\begin{aligned}\int_{\mathcal{C}_\Omega} y^{1-\alpha}\langle w_\varepsilon\nabla\phi, \phi\nabla w_\varepsilon\rangle dxdy &\le C\int_{C_r} y^{1-\alpha}|w_\varepsilon(x,y)||\nabla w_\varepsilon(x,y)|dxdy \\ &= C\varepsilon^{-N+\alpha-1}\int_{C_r} y^{1-\alpha}\left|w_{1,\alpha}\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)\right|\left|\nabla w_{1,\alpha}\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)\right|dxdy \\ &= C\varepsilon\int_{C_{\frac{r}{\varepsilon}}} y^{1-\alpha}|w_{1,\alpha}(x,y)|\,|\nabla w_{1,\alpha}(x,y)|dxdy.\end{aligned} \tag{3.45}$$

Moreover, for $(x,y) \in C_{r/\varepsilon}$ and $\alpha > 0$, we obtain that

$$\begin{aligned}w_{1,\alpha}(x,y) &= \int_{|z|<\frac{1}{4\varepsilon}} P_y^\alpha(x-z)u_{1,\alpha}(z)dz + \int_{|z|>\frac{1}{4\varepsilon}} P_y^\alpha(x-z)u_{1,\alpha}(z)dz \\ &\le C\varepsilon^{N+\alpha}y^\alpha\int_{|z|<\frac{1}{4\varepsilon}}\frac{dz}{|z|^{N-\alpha}} + C\varepsilon^{N-\alpha}\int_{\mathbb{R}^N} P_y^\alpha(z)dz \\ &\le Cy^\alpha\varepsilon^N + C\varepsilon^{N-\alpha} \le C\varepsilon^{N-\alpha}.\end{aligned} \tag{3.46}$$

If $\alpha < 1$, from (3.38), (3.45) and (3.46), it follows that

$$\int_{\mathcal{C}_\Omega} y^{1-\alpha}\langle w_\varepsilon\nabla\phi, \phi\nabla w_\varepsilon\rangle dxdy \le C\varepsilon^{1+2(N-\alpha)}\int_{C_{\frac{r}{\varepsilon}}} y^{-\alpha}dxdy = O(\varepsilon^{N-\alpha}). \tag{3.47}$$

To obtain the similar estimate for $\alpha > 1$ we use (3.39). Indeed by this estimate, together with (3.45) and (3.46) we get that

$$\int_{\mathcal{C}_\Omega} y^{1-\alpha}\langle w_\varepsilon\nabla\phi, \phi\nabla w_\varepsilon\rangle dxdy \le C\varepsilon^{2(1+N-\alpha)}\int_{C_{\frac{r}{\varepsilon}}} y^{1-\alpha}dxdy = O(\varepsilon^{N-\alpha}). \tag{3.48}$$

Note that for $\alpha = 1$, as w_ε is explicit, we can obtain the same estimate directly.

Then we have proved that

$$\|\phi w_\varepsilon\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} = \|w_\varepsilon\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} + O(\varepsilon^{N-\alpha}).$$

We now show that (3.41) holds.

$$\begin{aligned}\|\phi u_\varepsilon\|^2_{L^2(\Omega)} &= \int_\Omega \phi^2(x)\frac{\varepsilon^{N-\alpha}}{(|x|^2+\varepsilon^2)^{N-\alpha}}dx \\ &\ge \int_{\{|x|<r/2\}}\frac{\varepsilon^{N-\alpha}}{(|x|^2+\varepsilon^2)^{N-\alpha}}dx \\ &\ge \int_{\{|x|<\varepsilon\}}\frac{\varepsilon^{N-\alpha}}{(2\varepsilon^2)^{N-\alpha}}dx + \int_{\{\varepsilon<|x|<r/2\}}\frac{\varepsilon^{N-\alpha}}{(2|x|^2)^{N-\alpha}}dx\end{aligned}$$

$$= \quad C\varepsilon^\alpha + C\varepsilon^{N-\alpha}\int_\varepsilon^{r/2} \theta^{2\alpha-1-N} d\theta.$$

Finally, (3.42) follows in a similar way to (3.41), so we omit the details. □

With the above properties in mind, we define the family of functions $\eta_\varepsilon = \frac{\phi w_\varepsilon}{\|\phi u_\varepsilon\|_{L^{2^*_\alpha}(\Omega)}}$.

Lemma 3.9. *There exists $\varepsilon > 0$ small enough such that*

$$\sup_{t\geq 0} \widetilde{J}(t\eta_\varepsilon) < c^*. \tag{3.49}$$

Proof. Assume $N \geq 2\alpha$, we make use of the following estimate

$$(a+b)^p \geq a^p + b^p + \mu a^{p-1}b, \quad a, b \geq 0, \quad p > 1, \quad \text{for some}\, \mu > 0. \tag{3.50}$$

Therefore

$$G(w) \geq \frac{1}{2^*_\alpha} w^{2^*_\alpha} + \frac{\mu}{2} w^2 w_0^{2^*_\alpha - 2} \tag{3.51}$$

which implies

$$\widetilde{J}(t\eta_\varepsilon) \leq \frac{t^2}{2}\|\eta_\varepsilon\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} - \frac{t^{2^*_\alpha}}{2^*_\alpha} - \frac{t^2}{2}\mu \int_\Omega w_0^{2^*_\alpha-2}\eta_\varepsilon^2 dx.$$

Since $w_0 \geq a_0 > 0$ in $\operatorname{supp}(\eta_\varepsilon)$ we get

$$\widetilde{J}(t\eta_\varepsilon) \leq \frac{t^2}{2}\|\eta_\varepsilon\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} - \frac{t^{2^*_\alpha}}{2^*_\alpha} - \frac{t^2}{2}\widetilde{\mu}\|\eta_\varepsilon\|^2_{L^2(\Omega)} =: g(t).$$

It is clear that $\lim\limits_{t\to\infty} g(t) = -\infty$, and $\sup\limits_{t\geq 0} g(t)$ is attained at some $t_\varepsilon > 0$. By differentiating the above function we obtain

$$0 = g'(t_\varepsilon) = t_\varepsilon\|\eta_\varepsilon\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} - t_\varepsilon^{2^*_\alpha-1} - t_\varepsilon\widetilde{\mu}\|\eta_\varepsilon\|^2_{L^2(\Omega)}, \tag{3.52}$$

which implies

$$t_\varepsilon \leq \|\eta_\varepsilon\|^{\frac{2}{2^*_\alpha-2}}_{X_0^\alpha(\mathcal{C}_\Omega)}.$$

Observe that by Lemma 3.8 we have $t_\varepsilon \geq C > 0$. On the other hand, the function

$$t \mapsto \frac{t^2}{2}\|\eta_\varepsilon\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} - \frac{t^{2^*_\alpha}}{2^*_\alpha}$$

is increasing on $[0, \|\eta_\varepsilon\|^{\frac{2}{2^*_\alpha-2}}_{X_0^\alpha(\mathcal{C}_\Omega)}]$. Whence

$$\sup_{t\geq 0} g(t) = g(t_\varepsilon) \leq \frac{\alpha}{2N}\|\eta_\varepsilon\|^{\frac{2N}{\alpha}}_{X_0^\alpha(\mathcal{C}_\Omega)} - C\|\eta_\varepsilon\|^2_{L^2(\Omega)}.$$

Since $\|u_\varepsilon\|_{L^{2^*_\alpha}(\Omega)}$ is independent of ε, by Lemma 3.8 we have

$$\|\eta_\varepsilon\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} = \kappa_\alpha S(\alpha, N) + O(\varepsilon^{N-\alpha}) \tag{3.53}$$

and

$$\|\eta_\varepsilon\|^2_{L^2(\Omega)} = \begin{cases} O(\varepsilon^\alpha) & \text{if } N > 2\alpha, \\ O(\varepsilon^\alpha \log(1/\varepsilon)) & \text{if } N = 2\alpha. \end{cases}$$

Therefore, for $N > 2\alpha$, we get that

$$g(t_\varepsilon) \leq \frac{\alpha}{2N}(\kappa_\alpha S(\alpha, N))^{\frac{N}{\alpha}} + C\varepsilon^{N-\alpha} - C\varepsilon^\alpha < \frac{\alpha}{2N}(\kappa_\alpha S(\alpha, N))^{\frac{N}{\alpha}} = c^*. \tag{3.54}$$

If $N = 2\alpha$ the same conclusion follows.

The last case $\alpha < N < 2\alpha$ follows by using the estimate (3.50) which gives

$$G(w) \geq \frac{1}{2^*_\alpha} w^{2^*_\alpha} + w_0 w^{2^*_\alpha - 1}. \tag{3.55}$$

Then (3.55) jointly with (3.42) and arguing in a similar way as above finish the proof. □

Proof of Theorem 1.1-(3).

To finish the last statement in Theorem 1.1, in view of the previous results, we seek for critical values below level c^*. For that purpose, we want to use the classical MP Theorem by Ambrosetti-Rabinowitz in [3]. We define

$$\Gamma_\varepsilon = \{\gamma \in \mathcal{C}([0,1], X_0^\alpha(\mathcal{C}_\Omega)) : \ \gamma(0) = 0, \, \gamma(1) = t_\varepsilon\eta_\varepsilon\}$$

for some $t_\varepsilon > 0$ such that $\widetilde{J}(t_\varepsilon \eta_\varepsilon) < 0$. And consider the minimax value

$$c_\varepsilon = \inf_{\gamma \in \Gamma_\varepsilon} \max\{\widetilde{J}(\gamma(t)) : 0 \le t \le 1\}.$$

According to Lemma 3.4, $c_\varepsilon \ge 0$. By Lemma 3.9, for $\varepsilon \ll 1$,

$$c_\varepsilon \le \sup_{t \ge 0} \widetilde{J}(t\eta_\varepsilon) < c^* = \frac{\alpha}{2N} \left(\kappa_\alpha S(\alpha, N)\right)^{N/\alpha}.$$

This estimate jointly with Lemma 3.5 and the MPT [3] if the minimax energy level is positive, or the refinement of the MPT [19] if the minimax level is zero, give the existence of a second solution to $(P)_\lambda$. □

4. Linear and superlinear cases.

4.1. **Linear case.** The proof of Theorem 1.2 follows the ideas of [9]. Note that for $\alpha = 1$, where the minimizers given in (3.37) are explicit, this result was recently proved in [23].
The first part of that theorem is an straightforward calculus.

Proof of Theorem 1.2 (1). Let φ_1 be the first eigenfunction of $(-\Delta)^{\alpha/2}$ in Ω. We have

$$\int_\Omega (-\Delta)^{\alpha/4} u (-\Delta)^{\alpha/4} \varphi_1 \, dx = \int_\Omega \lambda_1 u \varphi_1 \, dx.$$

On the other hand,

$$\int_\Omega (-\Delta)^{\alpha/4} u (-\Delta)^{\alpha/4} \varphi_1 \, dx = \int_\Omega [u^{2^*_\alpha - 1} + \lambda u] \varphi_1 \, dx > \int_\Omega \lambda u \varphi_1 \, dx.$$

This clearly implies $\lambda < \lambda_1$. □

To prove the second part of Theorem 1.2 some notation is in order. We consider the following Rayleigh quotient

$$Q_\lambda(w) = \frac{\|w\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} - \lambda \|u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^{2^*_\alpha}(\Omega)}}$$

and

$$S_\lambda = \inf\{Q_\lambda(w) \mid \ w \in X_0^\alpha(\mathcal{C}_\Omega)\}. \tag{4.1}$$

Proposition 4.1. *Assume* $0 < \lambda < \lambda_1$. *Then* $S_\lambda < \kappa_\alpha S(\alpha, N)$.

Proof. Let $\phi = \phi_r$ be a cut-off function like in Lemma 3.8 and denote $\phi(x) := \phi(x, 0)$. Taking r sufficiently small we can use $\phi w_\varepsilon \in X_0^\alpha(\mathcal{C}_\Omega)$ as a test function in Q_λ, where w_ε is defined in (3.37). Denoting $K_1 = \|u_\varepsilon\|^{2^*_\alpha}_{L^{2^*_\alpha}(\Omega)}$, as before, K_1 is independent of ε, and moreover

$$\begin{aligned}
\int_\Omega |\phi u_\varepsilon|^{2^*_\alpha} dx &= \int_{\mathbb{R}^N} |\phi u_\varepsilon|^{2^*_\alpha} dx \\
&\ge \int_{|x| < r/2} |u_\varepsilon|^{2^*_\alpha} dx \\
&= K_1 - \int_{|x| > r/2} |u_\varepsilon|^{2^*_\alpha} dx \\
&\ge K_1 + O(\varepsilon^N).
\end{aligned} \tag{4.2}$$

Since w_ε is a minimizer of $S(\alpha, N)$, we have that

$$K_1^{-2/2^*_\alpha} \int_{\mathbb{R}^{N+1}_+} y^{1-\alpha} |\nabla w_\varepsilon|^2 \, dxdy = S(\alpha, N). \tag{4.3}$$

Finally, by (4.2) and using the estimates (3.40) and (3.41), for $N > 2\alpha$, we obtain that

$$Q_\lambda(\phi w_\varepsilon) \le \frac{\kappa_\alpha \displaystyle\int_{\mathbb{R}^{N+1}_+} y^{1-\alpha} |\nabla w_\varepsilon|^2 \, dxdy - \lambda C \varepsilon^\alpha + O(\varepsilon^{N-\alpha})}{K_1^{2/2^*_\alpha} + O(\varepsilon^N)}.$$

Therefore taking ε small enough, we get

$$Q_\lambda(\phi w_\varepsilon) \le \frac{\kappa_\alpha S(\alpha, N) - \lambda C \varepsilon^\alpha K_1^{-2/2^*_\alpha} + O(\varepsilon^{N-\alpha})}{1 + O(\varepsilon^N)}$$

$$\begin{aligned} &\leq \kappa_\alpha S(\alpha,N) - \lambda C\varepsilon^\alpha K_1^{-2/2^*_\alpha} + O(\varepsilon^{N-\alpha}) \\ &< \kappa_\alpha S(\alpha,N). \end{aligned}$$

On the other hand, a similar calculus for the case $N = 2\alpha$, proves that for ε small enough,

$$Q_\lambda(\phi w_\varepsilon) \leq \kappa_\alpha S(\alpha,N) - \lambda C\varepsilon^\alpha \log(1/\varepsilon) K_1^{-2/2^*_\alpha} + O(\varepsilon^\alpha) < \kappa_\alpha S(\alpha,N),$$

which finishes the proof. □

Recall now the Brezis-Lieb Lemma,

Lemma 4.1 ([6]). *Let Ω be an open set and $\{u_n\}$ be a sequence weakly convergent in $L^q(\Omega), 2 \leq q < \infty$ and a.e. convergent in Ω. Then* $\lim_{n\to\infty}(\|u_n\|^q_{L^q(\Omega)} - \|u_n - u\|^q_{L^q(\Omega)}) = \|u\|^q_{L^q(\Omega)}$.

This property allows us to we prove the following one.

Proposition 4.2. *Assume $0 < \lambda < \lambda_1$. Then the infimum S_λ defined in* (4.1) *is achieved.*

Proof. First, since $\lambda < \lambda_1$ we have that $S_\lambda > 0$. Let us take a minimizing sequence of S_λ, $\{w_m\} \subset X_0^\alpha(\mathcal{C}_\Omega)$ such that, without loss of generality, $w_m \geq 0$ and $\|w_m(\cdot,0)\|_{L^{2^*_\alpha}(\Omega)} = 1$. Clearly this implies that $\|w_m\|_{X_0^\alpha(\mathcal{C}_\Omega)} \leq C$, then there exists a subsequence (still denoted by $\{w_m\}$) verifying

$$\begin{array}{rcll} w_m & \rightharpoonup & w & \text{weakly in } X_0^\alpha(\mathcal{C}_\Omega), \\ w_m(\cdot,0) & \to & w(\cdot,0) & \text{strongly in } L^q(\Omega),\ 1 \leq q < 2^*_\alpha, \\ w_m(\cdot,0) & \to & w(\cdot,0) & \text{a.e in } \Omega. \end{array} \tag{4.4}$$

A simple calculation, using the weak convergence, gives that

$$\begin{aligned} \|w_m\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} &= \|w_m - w\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} + \|w\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} + 2\kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}\langle \nabla w, \nabla w_m - \nabla w\rangle dxdy \\ &= \|w_m - w\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} + \|w\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} + o(1). \end{aligned}$$

By Lemma 4.1, we have that $\|(w_m - w)(\cdot,0)\|_{L^{2^*_\alpha}(\Omega)} \leq 1$ for m big enough. Hence

$$\begin{aligned} Q_\lambda(w_m) &= \|w_m\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} - \lambda\|w_m(\cdot,0)\|^2_{L^2(\Omega)} \\ &= \|w_m - w\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} + \|w\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} - \lambda\|w_m(\cdot,0)\|^2_{L^2(\Omega)} + o(1) \\ &\geq \kappa_\alpha S(\alpha,N)\|(w_m - w)(\cdot,0)\|^2_{L^{2^*_\alpha}(\Omega)} + S_\lambda\|w(\cdot,0)\|^2_{L^{2^*_\alpha}(\Omega)} + o(1) \\ &\geq \kappa_\alpha S(\alpha,N)\|(w_m - w)(\cdot,0)\|^{2^*_\alpha}_{L^{2^*_\alpha}(\Omega)} + S_\lambda\|w(\cdot,0)\|^{2^*_\alpha}_{L^{2^*_\alpha}(\Omega)} + o(1). \end{aligned}$$

By Lemma 4.1 again, this leads to

$$\begin{aligned} Q_\lambda(w_m) &\geq (\kappa_\alpha S(\alpha,N) - S_\lambda)\|(w_m - w)(\cdot,0)\|^{2^*_\alpha}_{L^{2^*_\alpha}(\Omega)} + S_\lambda\|w_m(\cdot,0)\|^{2^*_\alpha}_{L^{2^*_\alpha}(\Omega)} + o(1) \\ &= (\kappa_\alpha S(\alpha,N) - S_\lambda)\|(w_m - w)(\cdot,0)\|^{2^*_\alpha}_{L^{2^*_\alpha}(\Omega)} + S_\lambda + o(1). \end{aligned}$$

Since $\{w_m\}$ is a minimizing sequence for S_λ, we obtain:

$$o(1) + S_\lambda \geq (\kappa_\alpha S(\alpha,N) - S_\lambda)\|(w_m - w)(\cdot,0)\|^{2^*_\alpha}_{L^{2^*_\alpha}(\Omega)} + S_\lambda + o(1).$$

Thus by Proposition 4.1

$$w_m(\cdot,0) \to w(\cdot,0) \qquad \text{in } L^{2^*_\alpha}(\Omega).$$

Finally, by a standard lower semi-continuity argument, w is a minimizer for Q_λ. □

Proof of Theorem 1.2 (2). By Proposition 4.2 there exists an α-harmonic function $w \in X_0^\alpha(\mathcal{C}_\Omega)$, such that $\|u\|^2_{L^{2^*_\alpha}(\Omega)} = 1$ and

$$\|w\|^2_{X_0^\alpha(\mathcal{C}_\Omega)} - \lambda\|u\|^2_{L^2(\Omega)} = S_\lambda$$

where $u = w(\cdot,0)$. Without loss of generality we may assume $w \geq 0$ (otherwise we take $|w|$ instead of w). So we get a positive solution of (P_λ). □

4.2. **Superlinear case.** In order to prove Theorem 1.3, the only difficult part is to show that we have a $(\mathrm{PS})_c$ sequence under the critical level $c = c^*$. This follows the same type of computations like in Lemma 3.9, with the estimate $\|\eta_\varepsilon\|^{q+1}_{L^{q+1}(\Omega)} \geq C\varepsilon^{\frac{\alpha-N}{2}q + \frac{\alpha+N}{2}}$ which holds for $N > \alpha(1+\frac{1}{q})$. In this case there is no limitation on $\lambda > 0$. We omit the complete details.

5. Regularity & Concentration-Compactness

We begin this section with some results about the boundedness and regularity of solutions. The next proposition is a refinement of Proposition 5.3 of [5] in order to cover the critical case $p = 2^*_\alpha - 1$. It is essentially based on [8].

Proposition 5.1. *Let $u \in H_0^{\alpha/2}(\Omega)$ be a solution to the problem*

$$\left\{\begin{array}{ll} (-\Delta)^{\alpha/2} u = f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{array}\right. \tag{5.1}$$

with f satisfying

$$0 \le f(x,s) \le C(1+|s|^p) \quad \forall\,(x,s) \in \Omega\times\mathbb{R}, \text{ and some } 0 < p \le 2^*_\alpha - 1. \tag{5.2}$$

Then $u \in L^\infty(\Omega)$ with $\|u\|_{L^\infty(\Omega)} \le C(\|u\|_{H_0^{\alpha/2}(\Omega)})$.

Proof. Let $w \in X_0^\alpha(\mathcal{C}_\Omega)$ be a solution to the problem

$$\left\{\begin{array}{ll} L_\alpha w = 0 & \text{in } \mathcal{C}_\Omega, \\ \dfrac{\partial w}{\partial \nu^\alpha} = f(\cdot, w) & \text{in } \Omega, \\ w = 0 & \text{on } \partial_L \mathcal{C}_\Omega. \end{array}\right. \tag{5.3}$$

Then $u = w(\cdot, 0)$ is a solution to (5.1). Let

$$a(x) := \frac{f(x,u)}{1+u(x)}.$$

Clearly

$$0 \le a \le C(1+u^{p-1}) \in L^{\frac{N}{\alpha}}(\Omega), \quad \text{for}\, 0 < p \le 2^*_\alpha - 1. \tag{5.4}$$

Given $T > 0$ we denote

$$w_T = w - (w-T)_+, \quad u_T = w_T(\cdot, 0).$$

For $\beta \ge 0$ we have

$$\begin{aligned} \|w w_T^\beta\|^2_{X_0^\alpha(C_\Omega)} = {} & \kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha} w_T^{2\beta} |\nabla w|^2 \, dxdy \\ & + \kappa_\alpha (2\beta + \beta^2) \int_{\{w \le T\}} y^{1-\alpha} w^{2\beta} |\nabla w|^2 \, dxdy. \end{aligned}$$

Using $\varphi = w w_T^{2\beta} \in X_0^\alpha(C_\Omega)$ as a test function we obtain

$$\kappa_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha} \langle \nabla w, \nabla(w w_T^{2\beta})\rangle \, dxdy = \int_\Omega f(u) u u_T^{2\beta} \, dx \le 2 \int_\Omega a(1+u^2) u_T^{2\beta} \, dx.$$

On the other hand, it is clear that

$$\begin{aligned} & \int_{\mathcal{C}_\Omega} y^{1-\alpha} \langle \nabla w, \nabla(w w_T^{2\beta})\rangle \, dxdy = \int_{\mathcal{C}_\Omega} y^{1-\alpha} w_T^{2\beta} |\nabla w|^2 \, dxdy + \\ & +2\beta \int_{\{w \le T\}} y^{1-\alpha} w^{2\beta} |\nabla w|^2 \, dxdy. \end{aligned}$$

Summing up, we have

$$\|w w_T^\beta\|^2_{X_0^\alpha(C_\Omega)} \le C \int_\Omega a(1+u^2) u_T^{2\beta} \, dx,$$

which by (2.11) implies that

$$\|u u_T^\beta\|^2_{L^{2^*_\alpha}(\Omega)} \le \widetilde{C} \int_\Omega a(1+u^2) u_T^{2\beta} \, dx, \tag{5.5}$$

with $\widetilde{C}$ some positive constant depending on α, β, N and $|\Omega|$. To compute the term on the right-hand side we add the hypothesis $u^{\beta+1}\in L^2(\Omega)$. With this assumption we get

$$\begin{aligned}\int_\Omega au^2u_T^{2\beta}\,dx &\le T_0\int_{\{a<T_0\}}u^2u_T^{2\beta}\,dx+\int_{\{a\ge T_0\}}au^2u_T^{2\beta}\,dx\\ &\le C_1T_0+\left(\int_{\{a\ge T_0\}}a^{\frac{N}{\alpha}}\,dx\right)^{\frac{\alpha}{N}}\left(\int_\Omega(uu_T^{\beta})^{2^*_\alpha}\,dx\right)^{\frac{2}{2^*_\alpha}}.\end{aligned}$$

By the same calculation,

$$\int_\Omega au_T^{2\beta}\,dx\le C_2T_0+\left(\int_{\{a\ge T_0\}}a^{\frac{N}{\alpha}}\,dx\right)^{\frac{\alpha}{N}}\left(\int_\Omega(u_T^{\beta})^{2^*_\alpha}\,dx\right)^{\frac{2}{2^*_\alpha}},$$

where, since $u^{\beta+1}\in L^2(\Omega)$, C_1 and C_2 can be taken independent of T. Hence, by (5.4) it follows that

$$\epsilon(T_0)=\left(\int_{\{a\ge T_0\}}a^{\frac{N}{\alpha}}\,dx\right)^{\frac{\alpha}{N}}\to 0\quad \text{as } T_0\to\infty.$$

Therefore, choosing T_0 large enough such that $C\epsilon(T_0)<\frac12$, by (5.5), we obtain that there exists a constant $K(T_0)$, independent of T, for which it holds

$$\|uu_T^{\beta}\|^2_{L^{2^*_\alpha}(\Omega)}\le K(T_0).$$

Letting $T\to\infty$ we conclude that $u^{\beta+1}\in L^{2^*_\alpha}(\Omega)$. Clearly we can obtain that $f(\cdot,u)\in L^r(\Omega)$ for some $r>N/\alpha$, in a finite number of steps. Thus, we conclude applying Theorem 4.7 of [5]. □

Now we characterize the regularity of the solutions of (P_λ) for the whole range of exponents.

Proposition 5.2. *Let u be a solution of (P_λ). Then the following hold*

(i) *If $\alpha=1$ and $q\ge 1$ then $u\in\mathcal{C}^\infty(\overline{\Omega})$.*
(ii) *If $\alpha=1$ and $q<1$ then $u\in\mathcal{C}^{1,q}(\overline{\Omega})$.*
(iii) *If $\alpha<1$ then $u\in\mathcal{C}^{\alpha}(\overline{\Omega})$.*
(iv) *If $\alpha>1$ then $u\in\mathcal{C}^{1,\alpha-1}(\overline{\Omega})$.*

Proof. First we observe that, by Proposition 5.1, we have $u\in L^\infty(\Omega)$ and also $f_\lambda(u)\in L^\infty(\Omega)$.

(i) Applying Proposition 3.1 of [12], we get that $u\in\mathcal{C}^\gamma(\overline{\Omega})$, for some $\gamma<1$. Since $q\ge 1$ then $f_\lambda(u)\in\mathcal{C}^\gamma(\overline{\Omega})$, so, again by Proposition 3.1 of [12], it follows that $u\in\mathcal{C}^{1,\gamma}(\overline{\Omega})$. Iterating the process we conclude that $u\in\mathcal{C}^\infty(\overline{\Omega})$.
(ii) As before we have $u\in\mathcal{C}^\gamma(\overline{\Omega})$, for some $\gamma<1$. Therefore $f_\lambda(u)\in\mathcal{C}^{q\gamma}(\overline{\Omega})$. It follows that $u\in\mathcal{C}^{1,q\gamma}(\overline{\Omega})$, which gives $f_\lambda(u)\in\mathcal{C}^{q}(\overline{\Omega})$. Finally this implies $u\in\mathcal{C}^{1,q}(\overline{\Omega})$.
(iii) By Lemma 2.8 of [14] we obtain that $u\in\mathcal{C}^\gamma(\overline{\Omega})$ for all $\gamma\in(0,\alpha)$. This implies that $f_\lambda(u)\in\mathcal{C}^r(\overline{\Omega})$ for every $r<\min\{q\alpha,\alpha\}$. Therefore, again by [14], this time using Lemmas 2.7 and 2.9, we get that $u\in\mathcal{C}^\alpha(\overline{\Omega})$.
(iv) Since $\alpha>1$, we can write problem (P_λ) as follows

$$\left\{\begin{array}{ll}(-\Delta)^{1/2}u=s & \text{in }\Omega,\\ (-\Delta)^{(\alpha-1)/2}s=f_\lambda(u) & \text{in }\Omega,\\ u=s=0 & \text{on }\partial\Omega.\end{array}\right.\tag{5.6}$$

Reasoning as before, we obtain the desired regularity in two steps, using Proposition 3.1 in [12] and Lemmas 2.7 and 2.9 in [14].

□

We end this section adapting to our setting a concentration-compactness result by P.L. Lions [20], used in the proof of Lema 3.5. We recall that a related concentration-compactness result for the fractional Laplacian has been recently obtained in [21]. Nevertheless, we need the version corresponding to the extended problem, and it cannot be deduced from the one in [21].

Theorem 5.1. *Let $\{w_n\}_{n\in\mathbb{N}}$ be a weakly convergent sequence to w in $X_0^\alpha(\mathcal{C}_\Omega)$, such that the sequence $\{y^{1-\alpha}|\nabla w_n|^2\}_{n\in\mathbb{N}}$ is tight. Let $u_n=Tr(w_n)$ and $u=Tr(w)$. Let μ, ν be two non negative measures such that*

$$y^{1-\alpha}|\nabla w_n|^2\to\mu\qquad \textit{and}\qquad |u_n|^{2^*_\alpha}\to\nu,\quad \textit{as } n\to\infty\tag{5.7}$$

in the sense of measures. Then there exist an at most countable set I and points $\{x_i\}_{i\in I}\subset\Omega$ such that

(1) $\nu=|u|^{2^*_\alpha}+\sum_{k\in I}\nu_k\delta_{x_k}$, $\nu_k>0$,

(2) $\mu\geq y^{1-\alpha}|\nabla w|^2+\sum_{k\in I}\mu_k\delta_{x_k}$, $\mu_k>0$,

(3) $\mu_k\geq S(\alpha,N)\nu_k^{\frac{2}{2^*_\alpha}}$.

Proof. Let $\varphi\in C_0^\infty(\overline{\mathcal{C}_\Omega})$. By the trace inequality (2.11) with $r=2^*_\alpha$ it follows that

$$S(\alpha,N)\left(\int_\Omega|\varphi w_n|^{2^*_\alpha}dx\right)^{2/2^*_\alpha}\leq\int_{\mathcal{C}_\Omega}y^{1-\alpha}|\nabla(\varphi w_n)|^2dxdy. \tag{5.8}$$

Let $K^*:=K_1\times K_2\subseteq\overline{\mathcal{C}}_\Omega$ be the support of φ and suppose first that the weak limit $w=0$. Then we get that

$$\begin{aligned}\int_{\mathcal{C}_\Omega}y^{1-\alpha}|\nabla(\varphi w_n)|^2dxdy &= \int_{K^*}y^{1-\alpha}|\nabla(\varphi w_n)|^2dxdy\\ &= \int_{K^*}y^{1-\alpha}|w_n|^2|\nabla\varphi|^2dxdy+\int_{K^*}y^{1-\alpha}|\varphi|^2|\nabla w_n|^2dxdy\\ &\quad +2\int_{K^*}y^{1-\alpha}w_n\varphi\langle\nabla\varphi,\nabla w_n\rangle dxdy.\end{aligned}$$

Since K^* is a bounded domain, and $y^{1-\alpha}$ is an A_2 weight, we have the compact inclusion

$$H^1(K^*,y^{1-\alpha})\hookrightarrow\hookrightarrow L^r(K^*,y^{1-\alpha})\,,\ 1\leq r<\frac{2(N+1)}{N-1}\,,\ \alpha\in(0,2).$$

Therefore, for a suitable subsequence, we get the limit

$$\int_{K^*}y^{1-\alpha}|w_n|^2|\nabla\varphi|^2dxdy\to0,\quad \text{as }\ n\to\infty.$$

By the weak convergence, given by hypothesis, we obtain

$$\int_{K^*}y^{1-\alpha}w_n\varphi\langle\nabla\varphi,\nabla w_n\rangle dxdy\to0,\quad \text{as }\ n\to\infty.$$

Hence, by (5.7) we conclude that

$$\int_{\mathcal{C}_\Omega}y^{1-\alpha}|\nabla(\varphi w_n)|^2dxdy\to\int_{\mathcal{C}_\Omega}|\varphi(x,y)|^2d\mu,\quad \text{as }\ n\to\infty.$$

Then, from (5.8) we get

$$S(\alpha,N)\left(\int_\Omega|\varphi|^{2^*_\alpha}d\nu\right)^{2/2^*_\alpha}\leq\int_{\mathcal{C}_\Omega}|\varphi|^2d\mu,\quad\forall\,\varphi\in C_0^\infty(\overline{\mathcal{C}}_\Omega). \tag{5.9}$$

If now $w\neq0$, we apply the above result to the function $v_n=w_n-w$. Indeed if

$$y^{1-\alpha}|\nabla v_n|^2\to d\tilde\mu\qquad\text{and}\qquad|v_n(\cdot,0)|^{2^*_\alpha}\to d\tilde\nu,\quad\text{as }\ n\to\infty,$$

it follows that

$$S(\alpha,N)\left(\int_\Omega|\varphi|^{2^*_\alpha}d\tilde\nu\right)^{2/2^*_\alpha}\leq\int_{\mathcal{C}_\Omega}|\varphi|^2d\tilde\mu\,,\quad\forall\,\varphi\in C_0^\infty(\overline{\mathcal{C}}_\Omega),$$

therefore, ([20]), for some sequence of points $\{x_k\}_{k\in I}\subset\Omega$, we have

$$d\tilde\nu=\sum_{k\in I}\tilde\nu_k\delta_{x_k}\,,\qquad d\tilde\mu\geq\sum_{k\in I}\tilde\mu_k\delta_{x_k}\,,$$

with $\tilde\mu_k\geq S(\alpha,N)\tilde\nu_k^{2^*_\alpha/2}$. Hence, by Lemma 4.1, we obtain

$$d\nu=|u|^{2^*_\alpha}+\sum_{k\in I}\tilde\nu_k\delta_{x_k}.$$

Let now φ be a test function. We have

$$\begin{aligned}\int_{\mathcal{C}_\Omega} y^{1-\alpha}\varphi|\nabla w_n|^2 dxdy &= \int_{\mathcal{C}_\Omega} y^{1-\alpha}\varphi|\nabla w|^2 dxdy + \int_{\mathcal{C}_\Omega} y^{1-\alpha}\varphi|\nabla (w_n-w)|^2 dxdy \\ &\quad +2\int_{\mathcal{C}_\Omega} y^{1-\alpha}\varphi\langle\nabla(w_n-w),\nabla w\rangle dxdy.\end{aligned}$$

Taking limits as $n\to\infty$ we get that

$$\begin{aligned}\int_{\mathcal{C}_\Omega}\varphi d\mu &= \int_{\mathcal{C}_\Omega} y^{1-\alpha}\varphi|\nabla w|^2 dxdy + \int_{\mathcal{C}_\Omega}\varphi d\tilde{\mu} \\ &\geq \int_{\mathcal{C}_\Omega} y^{1-\alpha}\varphi|\nabla w|^2 dxdy + \int_{\mathcal{C}_\Omega} y^{1-\alpha}\varphi\sum_{k\in I}\tilde{\mu}_k\delta_{x_k} dxdy,\end{aligned}$$

with the same condition $\tilde{\mu}_k \geq S(\alpha,N)\tilde{\nu}_k^{2^*_\alpha/2}$. So we obtain the desired conclusion. □

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